NUMERICAL INTEGRATION

Given a set of data points $(x_0, y_0), (x_1, y_1) - - - - - - , (x_n, y_n)$ of a function $y = f(x)$ where $f(x)$ is not known explicitly, it is required to compute the value of the definite integral

$$\boxed{I = \int_a^b y\,dx} \quad \text{——— (1).}$$

We derive a general formula for numerical integration using Newton's forward difference formula.

Let the interval $[a, b]$ be divided into n equal subintervals such that $a = x_0 < x_1 < - - - , x_n = b$. Clearly, $x_n = x_0 + nh$, where, $h = x_1 - x_0 = x_2 - x_1 - - - - - - x_n - x_{n-1}$.

Hence, the integral in equation (1) becomes

$$I = \int_{x_0}^{x_n} y\,dx.$$

Now, approximating y by Newton's forward difference formula, we obtain,

$$I = \int_{x_0}^{x_n} \left[y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0 + - - \right]dx$$

Since, $x = x_0 + ph$, then $dx = h\,dp$ and hence the above integration becomes

$$I = h\int_0^n \left[y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0 + - - \right]dp$$

[Explaination for the change of limit :

we have, $x = x_0 + ph$.

$$\Rightarrow p = \frac{x - x_0}{h}$$

Now, as $x \to x_0$, $p \to 0$.

and as $x \to x_n$, $p \to \frac{x_n - x_0}{h}$

i.e $\frac{x_0 + nh - x_0}{h}$

$\therefore p \to n$.

So, the new limits become 0 to n.

Now,
$$I = h\left[py_0 + \frac{p^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{p^3}{3} - \frac{p^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{p^4}{4} - 3\times\frac{p^3}{3} + 2\times\frac{p^2}{2}\right)\Delta^3 y_0 + \cdots\right]_0^n$$

$$= h\left[py_0 + \frac{p^2}{2}\Delta y_0 + \frac{p^2(2p-3)}{12}\Delta^2 y_0 + \frac{p^2(p^2-4p+4)}{24}\Delta^3 y_0 + \cdots\right]_0^n$$

$$= h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{n^2(2n-3)}{12}\Delta^2 y_0 + \frac{n^2(n^2-4n+4)}{24}\Delta^3 y_0 + \cdots\right]$$

$$= nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\right]$$

$$\therefore \boxed{\int_{x_0}^{x_n} y\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\right]} \quad \text{——— (2)}$$

From the above general formula, given in equation (2), we can obtain different integration formulae by putting $n = 1, 2, 3, \ldots$ etc.

(1) Trapezoidal Rule:

Setting $n = 1$ in the general formula given in equation (2), all differences higher than the first will become zero and we obtain,

$$\int_{x_0}^{x_1} y\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right] \qquad \left[\because \Delta y_0 = y_1 - y_0\right.$$

$$= h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right]$$

$$= h\left[y_0 + \frac{1}{2}y_1 - \frac{1}{2}y_0\right]$$

$$= h\left[\frac{1}{2}y_0 + \frac{1}{2}y_1\right]$$

$$= \frac{h}{2}(y_0 + y_1)$$

$$\therefore \boxed{\int_{x_0}^{x_1} y\,dx = \frac{h}{2}(y_0 + y_1)}$$

Similarly, for the next interval, $[x_1, x_2]$, we obtain

$$\boxed{\int_{x_1}^{x_2} y\,dx = \frac{h}{2}(y_1 + y_2)}$$ and so on.

For the last interval $[x_{n-1}, x_n]$, we have,

$$\boxed{\int_{x_{n-1}}^{x_n} y\,dx = \frac{h}{2}(y_{n-1} + y_n)}$$

Combining all the above expressions, we obtain the rule,

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}(y_0+y_1) + \frac{h}{2}(y_1+y_2) + \cdots \cdots + \frac{h}{2}(y_{n-1}+y_n).$$

$$\therefore \boxed{\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[y_0 + 2(y_1+y_2+\cdots+y_{n-1}) + y_n\right]} \quad \text{--- (3)}$$

This is known as the trapezoidal rule.

The total error of the trapezoidal formula is given by

$$\boxed{E_1 = -\frac{1}{12}h^3(y_0'' + y_1'' + \cdots + y_{n-1}'')} \quad \text{--- (4)}$$

Assuming that $y''(\bar{x})$ is the largest value of the n quantities on the right hand side of eq^n (4) we obtain,

$$E_1 = -\frac{1}{12}h^3 n y''(\bar{x}) = -\frac{b-a}{12}h^2 y''(\bar{x}) \quad \left[\because h = \frac{b-a}{n}\right.$$

(2) Simpson's 1/3 - Rule :

This rule is obtained by putting $\boxed{n=2}$ in eqn (2), and proceeding in the same manner as in the trapezoidal formula derivation, we get,

$$\boxed{\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\left[y_0 + 4(y_1+y_3+y_5+\cdots+y_{n-1}) + 2(y_2+y_4+\cdots+y_{n-2}) + y_n\right].}$$

which is the Simpson's 1/3 - rule or simply Simpson's rule.

The error in this formula is given by.

$$\boxed{E_2 = -\frac{b-a}{180} h^4 y^{iv}(\bar{x})}$$

where $y^{iv}(\bar{x})$ is the largest value of the fourth derivatives.

(3) Simpson's 3/8 - Rule :

Setting $n=3$ in equation (2) and proceeding the same manner as in the above two methods, we get,

$$\boxed{\int_{x_0}^{x_n} = \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + 2y_3 + 3y_4 + 3y_5 + 2y_6 + \cdots + \cdots 2y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right].}$$

which is the Simpson's 3/8 rule, it is not so accurate as Simpson's rule.

Example: Evaluate $I = \int_0^1 \frac{1}{1+x}\,dx$,

correct to three decimal places using
(i) trapezoidal rule (ii) Simpson's rule. Take $h = 0.5$.

Solution :— Here, $h = 0.5$.

$\therefore x_0 = 0,\ x_1 = 0.5,\ x_2 = 1.0$.

and $y_0 = 1.0000,\ y_1 = 0.6667,$
$y_2 = 0.5000$.

$\left[\ y = \frac{1}{1+x}.\ \text{at } x_0 = 0,\ y_0 = \frac{1}{1+0} = 1.\right.$

(i) Using Trapezoidal rule we have,

$$I = \frac{h}{2}\left[y_0 + 2y_1 + y_2\right].$$

$$= \frac{0.5}{2}\left[1.0000 + 2(0.6667) + 0.5000\right].$$

$$= 0.70835$$

(ii) Using Simpson's rule we have,

$$I = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right] = \frac{0.5}{3}\left[1.0000 + 4(0.6667) + 0.5\right].$$

$$= 0.6945.$$

To do: Evaluate the same integral i.e

$$I = \int_0^1 \frac{1}{1+x}\,dx$$ correct to three decimal places

using both trapezoidal and Simpson's rules, taking $h = 0.25$ and 0.125 respectively.